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EFFECTIVE PLASMA HEAT CONDUCTIVITY IN "BRAIDED" MAGNETIC FIELD—II. PERCOLATION LIMIT

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Abstract—This paper is devoted to the problem of anomalous transport across a magnetic field that includes a small stochastic component $\delta\mathbf{B}$. The perturbation is assumed to be so strongly stretched along the background magnetic field $\mathbf{B}_0$ that the parameter R is large: $R \equiv b_0 L_0/\delta \gg 1$ (here $b_0 \equiv \delta B_\perp/B_0 \ll 1$, and L_0 is the longitudinal and δ the transverse correlation length of the magnetic perturbation). This strong turbulence limit, which is opposite to the quasi-linear one ($R \ll 1$), has certain notable features. The principal result is that the main transport is concentrated in very thin regions, being fractal sets with the dimension d_f, which can range in value from 2 to 2.75, depending on the spectrum of the magnetic perturbation. These regions consist of a small fraction of magnetic lines that percolate, that is, walk from the non-perturbed magnetic flux surfaces to a distance large compared to the transverse correlation length δ. Due to such a strong inhomogeneity of the transport distribution, as well as the long correlations, the standard transport averaging techniques fail, and one should make use of the percolation theory methods. Thus the strong turbulence regime is referred to here as *the percolation limit*. In comparison with the quasi-linear limit, the percolation limit has several additional intermediate regimes and the expressions for the effective heat conductivity χ_{eff} include the critical exponents of 2-D percolation theory. The estimates of χ_{eff} are obtained both in the collisional and collisionless limits, including the case of non-stationary magnetic perturbations.

1. INTRODUCTION

THE PURPOSE of the present paper is to extend earlier works on quasi-linear cross-field stochastic transport (RECHESTER and ROSENBLUTH, 1978; KADOMTSEV and POGUTSE, 1979; KROMMES, 1978; KROMMES *et al.*, 1983; ISICHENKO, 1991) to the case of strong magnetic perturbations:

$$R \equiv b_0 L_0/\delta \gg 1, \tag{1}$$

where $b_0 \equiv \delta B_\perp/B_0 \ll 1$ is the relative magnitude of the magnetic field perturbation, and L_0 is the longitudinal and δ the transverse correlation length. In this paper, we adopt the notation introduced in Part I (ISICHENKO, 1991). The present part of our study has been written as a separate paper because the treatment of the problem in limit (1) uses new techniques related to the continuum percolation problem (GRUZINOV *et al.*, 1990). For this reason, we shall refer to limit (1) as *the percolation limit*.

Let us consider a stationary "braided" magnetic field

$$\mathbf{B} = B_0\hat{z} + \delta\mathbf{B}(x, y, z, t). \tag{2}$$

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In the present paper for simplicity we neglect shear effects. With equation (2), the equation of a field line takes the form

$$d\mathbf{r}_\perp/dz = \mathbf{b}(\mathbf{r}_\perp, z), \tag{3}$$

where $\mathbf{b} \equiv \delta\mathbf{B}_\perp/B_0$, $\mathbf{r}_\perp \equiv (x, y)$. Equation (3) describes nearly two-dimensional motion of a magnetic line, since inequality (1) suggests that the dependence of the RHS of equation (3) on z is very slow. Besides, in this approximation one can consider the transverse magnetic perturbation $\mathbf{b}$ to be incompressible. According to div $\delta\mathbf{B} = 0$ and (1), we have

$$\operatorname{div}\mathbf{b} = \partial b_x/\partial x + \partial b_y/\partial y = -\partial b_z/\partial z \approx b_z/L_0 \ll |\nabla \times \mathbf{b}| \approx b_0/\delta. \tag{4}$$

Equation (4) implies that the compressibility of $\mathbf{b}$ is insignificant. The criterion for the neglect of compressibility is discussed in more detail in Appendix A. Thus we can express $\mathbf{b}$ in terms of the longitudinal vector potential $\psi(x, y, z)$ which also depends on z very slowly:

$$\mathbf{b} = \nabla\psi \times \hat{z}. \tag{5}$$

Thus, to a first approximation, due to the large parameter R, every field line produces cylindrical screw-type revolutions around a surface of constant ψ. The transverse walk of a line is hence restricted to the size of the corresponding contour of $\psi(x, y, z)$, at a given coordinate $z = z_0$. For a turbulent state of a magnetized plasma one may assume a random distribution of ψ. For simplicity, we take the perturbation as statistically isotropic. Among the contours of a random function, most are closed on the correlation scale δ. However, there is a small portion of level lines that are much longer. For example, the Earth's relief exhibits not only lakes and islands, but also continental coastlines. In the limit $R \to \infty$ (exact 2-D case), the integral of motion $\psi(x, y)$, even though a random function, prevents stochastic spreading of magnetic lines. Yet, at large but finite R the magnetic transport should develop, beginning in the first turn from very large contours of $\psi(x, y, z_0)$ which provide "long-correlated jumps" of field lines. The importance of these large contours for plasma transport is due to their coherent contribution to the diffusion of magnetic lines.

So, under condition (1), the effective transport must be long-correlated, due to the important role of the transverse walk of magnetic lines to distances large compared to δ. For the treatment of the problem one must study the distribution of isolines of a random function of two variables over their sizes. This problem is closely connected with the percolation problem (cf. STAUFFER, 1979; SHKLOVSKII and EFROS, 1984). KADOMTSEV and POGUTSE (1979) were the first to point out the relevance of the percolation theory to the limit $R \gg 1$, but the cursory application of that theory in their work lead to an incorrect expression for the magnetic diffusion coefficient.

GRUZINOV *et al.* (1990) considered the problem of low-frequency turbulent diffusion in two dimensions [being exactly equivalent to the problem of diffusing magnetic lines described by equation (1) with $R \gg 1$]. Using recent analytical results in 2-D percolation theory (SALEUR and DUPLANTIER, 1987), they derived the following scaling of the turbulent diffusion coefficient:

$$D_m \approx b_0 \delta R^{-3/10}. \tag{6}$$

Equation (6) is written as the diffusivity of magnetic lines. The assessment (6) suggests that in the two-dimensional (integrable) case $R = \infty$ the magnetic diffusion D_m vanishes, which differs from the previously reported estimate of $D_m \approx b_0\delta$ (KADOMTSEV and POGUTSE, 1979 ; GALEEV and ZELENYI, 1981 ; KROMMES *et al.*, 1983).

In the limit $R \gg 1$ the exponentiation of adjacent lines is different from in the quasi-linear limit. In the percolation limit the role of Kolmogorov entropy in average transport is more complicated than in the quasi-linear limit, and requires a more subtle consideration. Specifically, in this paper it is shown that an appropriate test-particle decorrelation length is expressed through the length l_m of the convolution of a magnetic flux tube (defined in Appendix B), rather than that of the exponentiation of field lines. Nevertheless, these two processes still remain closely connected.

Perhaps the most striking feature of stochastic transport in the percolation limit is that the major part of the heat and particle fluxes is concentrated in very thin regions occupying an infinitesimally-small fraction of the plasma volume, and, due to their self-similarity, the regions can be described in terms of fractal geometry. At the same time, due to the previously-discussed role of large contours of ψ, the cross-field flux correlation function decays relatively slowly, up to an anomalously-large correlation, or mixing, length $a_m \gg \delta$. These features, on the one hand, leave no hope for applying standard averaging techniques and/or convergence of the perturbation series, and, simultaneously, make it extremely difficult to analyze the problem numerically. To study the geometry of a stochastic magnetic field in the limit $R \gg 1$, one is forced to employ a direct **x**-space non-perturbative formalism, like the percolation theory.

In other respects the solution of the effective heat conductivity problem in the percolation limit is based upon the same techniques as the quasi-linear approximation. In what follows, we will use test-particle motion analysis in order to obtain scaling laws for cross-field plasma transport, with particular emphasis on distinguishing physically-different transport regimes.

The remainder of the article is organized as follows. In Section 2, we discuss the diffusion and the exponentiation of magnetic lines. In Section 3, the effective perpendicular electron heat conductivity χ_{eff} is expressed through the test-particle decorrelation time t_d both for the hydrodynamic ($v_e t_d > 1$) and the kinetic ($v_e t_d < 1$) limits, which is intended to generalize the "double diffusion" theory of KROMMES *et al.* (1983). The very decorrelation time is evaluated in Sections 4 and 5, for stationary and non-stationary magnetic perturbations, respectively. In Section 6, we demonstrate the transition between various anomalous cross-field transport regimes and summarize the results obtained. Lengthy auxiliary arguments are outlined in the Appendices. In Appendix A we take into account the compressibility of the transverse component **b** of the magnetic perturbation in order to establish the limits of applicability of the incompressible approximation. In Appendix B we discuss the stochasticity of magnetic lines and its relation to the decorrelation of a test particle from a specified field line. Appendix C is devoted to the effective heat conductivity in a 2-D random magnetic field ($R = \infty$). For this case the application of the DYKHNE (1971) technique is discussed.

In the present paper we make use of the notation and approaches from Part I, thereby reducing the need for lengthy explanations.

2. PERCOLATION GEOMETRY OF STOCHASTIC MAGNETIC FIELDS

In this section we relate stochastic magnetic field lines to contours of a random function and discuss the application of the continuum percolation problem to the study of magnetic field line geometry. The results of GRUZINOV *et al.* (1990), which are given a short review, are applied.

As stated in Section 1, at $R \gg 1$ the (x, y)-projection of a magnetic line nearly follows the contours of the vector potential $\psi(x, y, z)$, with z considered as a slowly-varying parameter. Bearing in mind that in the limit involved large contours are of primary importance, one must preface the magnetic diffusion problem with the study of the statistics of random isolines.

The statistical topography of a random relief $\psi(x, y)$ is described by the continuum percolation problem (SHKLOVSKII and EFROS, 1984). Since this problem can be considered as a limiting case of a lattice percolation problem (see, for instance, the review by STAUFFER, 1979), and due to the universality of percolation-critical exponents (SYKES and ESSAM, 1964), the scaling of the contours' distribution function in the limit of large contour size can be determined analytically (GRUZINOV *et al.*, 1990). Let us briefly summarize the results of the paper that are relevant for our discussion:

(a) Suppose one can ascribe to the random and statistically-isotropic function $\psi(x, y)$ a single characteristic oscillatory amplitude $\psi_0 = b_0\delta$, and a single spatial (correlation) scale δ. Then the distribution function of the contours of ψ over their diameters a has the following long-range scaling:

$$F(a) \approx \delta/a, \quad a \gg \delta. \tag{7}$$

Here $F(a)$ implies the fraction of area occupied by contours with diameters (understood as maximum linear size) from a to $2a$. According to (7), most of the contours have sizes $a \approx \delta$. The space-average of a quantity A which depends on the diameter of a contour can be calculated with the help of equation (7) as

$$\langle A \rangle = \int_{\delta}^{\infty} F(a)A(a)\, \mathrm{d}a/a. \tag{8}$$

(b) Any sufficiently long contour with $a \gg \delta$, considered on the scale λ, $\delta \ll \lambda \ll a$, is a fractal (i.e. statistically self-similar) curve with the fractal dimension

$$d_h = (v+1)/v = 7/4, \tag{9}$$

where $v = 4/3$ is the correlation exponent of the 2-D percolation problem. In particular, the length L of a contour is much greater than its diameter a and scales as

$$L(a) \approx \delta(a/\delta)^{(v+1)/v}, \quad a \gg \delta. \tag{10}$$

(c) From (a) and (b) it follows that the set of contours with diameters of the order of a (say, a to $2a$) consists of densely-packed fractal cells (let us call them "a-type cells"—see Fig. 1), each of which looks like a web with the thread width

$$h(a) \approx a^2F(a)/L(a) \approx \delta(\delta/a)^{1/\nu}, \quad a \gg \delta. \tag{11}$$

In what follows, we will call the quantity $h(a)$ the width of the a-type cell. Below we will also refer to cells composed of the contours of the magnetic vector potential ψ as "the magnetic cells".

(d) Let the function ψ include a smooth dependence on a third parameter z, $\psi(x, y, z)$ being a random function of (x, y) at every fixed z. If the characteristic inhomogeneity (correlation) scale over z is L_0, then a-type cells are unrecognizably changed upon the displacement along z,

$$\delta z(a) \approx L_0 h(a)/\delta, \quad a \gg \delta, \tag{12}$$

which implies a perturbation corresponding to the thickness of the web.

The above results relate to a "single-scale" random function. Perhaps a more interesting case would deal with multiple scales characterized by, say, the power Fourier spectrum of $\psi(\mathbf{r})$. The problem of multiple-scale random topography has recently been solved by ISICHENKO and KALDA (1991) using an extended percolation approach. One of the peculiarities of that model is the fractal dimension d_h of long level lines which can take any value between 1 and 7/4, depending on the spectrum of ψ. For the sake of simplicity, the present paper is restricted to the widely-used single-scale model.

The geometric properties of random contours (a)–(d) are sufficient to calculate the magnetic diffusivity D_m and the Kolmogorov entropy in the percolation limit $R \gg 1$. The mixing length a_m, i.e. the size of a contour performing the most effective contribution to D_m, can be assessed as the maximum transverse correlated walk of a magnetic line. This corresponds to the case where the field-line projection performs a complete revolution around the contour, by the longitudinal displacement (12) result-

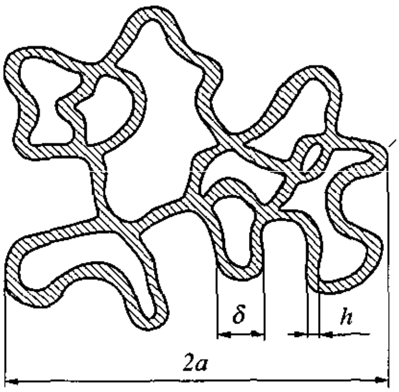

FIG. 1.—Percolation magnetic cell. The dashed area is occupied by contours of ψ with diameters $[a, 2a]$, $a \gg \delta$.

ing in the destruction of the magnetic cell via the reconnection of contours. So, we write

$$\delta z(a_{\mathrm{m}})b_0 = L(a_{\mathrm{m}}). \tag{13}$$

From equations (10)–(13) one readily obtains

$$a_{\mathrm{m}} = \delta R^{\nu/(\nu+2)}, \tag{14}$$

which corresponds to the longitudinal displacement

$$z_{\mathrm{m}} \equiv \delta z(a_{\mathrm{m}}) = L_0 R^{-1/(\nu+2)}, \tag{15}$$

and the magnetic cell width

$$h_{\mathrm{m}} \equiv h(a_{\mathrm{m}}) = \delta R^{-1/(\nu+2)}. \tag{16}$$

Now the diffusivity of the magnetic lines can be heuristically calculated as

$$D_{\mathrm{m}} = F(a_{\mathrm{m}})a_m^2/z_m = b_0 h_{\mathrm{m}} = b_0 \delta R^{-1/(\nu+2)}. \tag{17}$$

With the numerical value of the percolation exponent $\nu = 4/3$, equation (17) yields expression (6).

Rigorously speaking, to calculate D_{m}, one should average the magnetic diffusion over all possible scales with the help of (8), namely

$$D_{\mathrm{m}} = \int_{\delta}^{\infty} D_{\mathrm{m}}(a)F(a)\,\mathrm{d}a/a, \tag{18}$$

with the "partial" diffusivity

$$D_{\mathrm{m}}(a) = \frac{a_{\perp}^2(a)}{\delta z(a)}. \tag{19}$$

Here $a_{\perp}(a)$ denotes the transverse displacement of a magnetic line corresponding to the longitudinal walk $\delta z(a)$. At $a < a_{\mathrm{m}}$ a magnetic line performs many revolutions around its contour, thus giving

$$a_{\perp}(a) = a, \quad a < a_{\mathrm{m}}. \tag{20}$$

In the opposite case, $a > a_{\mathrm{m}}$, the line passes only a small part of the contour resulting in the displacement found from equation (10):

$$a_{\perp}(a) = \delta\left(\frac{\delta z(a)b_0}{\delta}\right)^{\nu/(\nu+1)}, \quad a > a_{\mathrm{m}}. \tag{21}$$

Combining equations (18)–(21) we simply calculate D_m to obtain the above result (17). This supports the conclusion that the major contribution to magnetic diffusion (i.e. of the order of 50%) is made by a small share of magnetic lines occupying the volume fraction

$$F(a_m) = R^{-\nu/(\nu+1)} \ll 1. \tag{22}$$

Another feature of magnetic transport in the percolation limit is its self-similar behavior in the inertial range of transverse scales $[\delta, a_m]$. This is connected with the fractal geometry of random contours (MANDELBROT, 1982). Let us introduce the concept of the effective transport region, denoting the region of minimum volume responsible for, say, 50% of the transport. Then in the inertial range $[\delta, a_m]$ of scales the effective transport region is a fractal, whose fractal dimension d_f can be calculated by adding unity to the fractal dimension of its plane cross-section (MANDELBROT, 1982), this being in our case the a_m-type magnetic cell. Hence, in the single-scale approximation, using (9) we have

$$d_f = d_h + 1 = 2.75. \tag{23}$$

In a more general case of multiple-scale magnetic turbulence $1 \leqslant d_h \leqslant 1.75$ (ISICHENKO and KALDA, 1991), hence

$$2 \leqslant d_f \leqslant 2.75. \tag{24}$$

Analogously to the quasi-linear limit, magnetic lines described by equation (1) in the percolation limit also exhibit stochastic exponentiation. However, this behavior is now strongly intermittent. A given couple of infinitesimally-close field lines diverge for a very long distance very slowly (namely, linearly with z), but then the distance between the lines increases abruptly up to a finite value of the order of δ. This effect is governed by the distribution of saddle (elliptic) points of $\psi(x, y, z_0)$ and it is more convenient to describe it in terms of the elongation of a curve being projected along the magnetic lines. In this representation the irregularities of the scattering of the magnetic lines are smeared out, and the curve undergoes an exponential anfractuous elongation with the growth rate estimated by GRUZINOV *et al.* (1990). For the length of the curve we have

$$\mathscr{L}(z) \approx \mathscr{L}(0) \exp(z/l), \quad l \approx \frac{L_0}{\sqrt{R} \log R}. \tag{25}$$

The inverse quantity of l could be regarded as the Kolmogorov, or topological entropy, of the case under consideration.

3. CONNECTION BETWEEN THE EFFECTIVE HEAT CONDUCTIVITY AND THE TIME OF DECORRELATION

In this section expressions for $\chi_{\rm eff}(t_{\rm d})$, both for the collisional ($\nu_{\rm c}t_{\rm d} > 1$) and collisionless ($\nu_{\rm c}t_{\rm d} < 1$) cases, are derived. The evaluation of the decorrelation time $t_{\rm d}$ is addressed in the next sections.

The effective diffusivity of a test particle, being the same to an order of magnitude as the effective heat conduction $\chi_{\rm eff}$, is defined through the square-average transverse displacement $\mathbf{r}_\perp$ of an electron at the decorrelation time:

$$\chi_{\rm eff} \approx \langle \mathbf{r}_\perp^2(t_{\rm d})\rangle / t_{\rm d}, \tag{26}$$

where the averaging is taken over the space of the initial conditions or, similarly, over the magnetic lines.

While moving along a magnetic line, which nearly traces out a spiral, the (x, y)-projection of the point passes the distance

$$L(z) \approx b_0 z. \tag{27}$$

If this path does not exceed the transverse correlation length δ, the transverse displacement is equal to $L(z)$, regardless of the magnetic line. For the percolating magnetic lines, and $L(z) \gg \delta$, the displacement $r_\perp(z)$ is defined by equation (10), but cannot exceed the diameter of the given magnetic line spiral a:

$$r_\perp(z) \approx \min\{\delta(L(z)/\delta)^{\nu/(\nu+1)}, a\}. \tag{28}$$

If we use equations (7) and (8), then for this case we obtain

$$\langle r_\perp^2(z)\rangle \approx \delta^2(L(z)/\delta)^{\nu/(\nu+1)}. \tag{29}$$

Expression (29) is valid until $r_\perp$ exceeds the mixing length (14), i.e. while $L(z) < L_{\rm m} = \delta \cdot R^{(\nu+1)/(\nu+2)}$, and after that the transverse walk of the magnetic line is a diffusion-like one with diffusivity (6): $\langle \mathbf{r}_\perp^2(z)\rangle \approx D_{\rm m}|z|$.

Summarizing what has been said above we derive the following expression for the effective transverse heat conduction:

$$\chi_{\rm eff} \approx \begin{cases} b_0^2 z_{\rm d}^2/t_{\rm d}, & z_{\rm d} < \delta/b_0, \quad \text{(QD)} \quad (30a) \\ (b_0 z_{\rm d}/\delta)^{\nu/(\nu+1)}\delta^2/t_{\rm d}, & \delta/b_0 < z_{\rm d} < z_{\rm m}, \quad \text{(IR)} \quad (30b) \\ D_{\rm m} z_{\rm d}/t_{\rm d}, & z_{\rm m} < z_{\rm d}. \quad \text{(MD)} \quad (30c) \end{cases}$$

Here $z_{\rm d}$ denotes the path the test particle takes along the magnetic field in the decorrelation time $t_{\rm d}$. Depending on the collision frequency $\nu_{\rm c}$, it is expressed as follows:

$$z_{\rm d}(t_{\rm d}) = \begin{cases} (\chi_\parallel t_{\rm d})^{1/2}, & \nu_{\rm c}t_{\rm d} > 1, \quad (31a) \\ v_{\rm e}t_{\rm d}, & \nu_{\rm c}t_{\rm d} < 1, \quad (31b) \end{cases}$$

which means hydrodynamic (collisional) and kinetic (collisionless) limits, respectively. The abbreviations used in (30) distinguish the quick decorrelation regime (QD), intermediate regime (IR), and the regime of magnetic line diffusion (MD). Equation (30) is similar to equation (18) of Part I derived for the quasi-linear limit, except for the appearance of a new intermediate regime lying between the QD and MD regimes.

When (31a) is substituted into equations (30a) and (30c), two expressions for χ_{eff} are obtained corresponding to the "fluid" ($\chi_{\text{eff}} = \chi_{\|} b_0^2$) and the "double diffusion" [$\chi_{\text{eff}} = D_{\text{m}}(\chi_{\|}/t_{\text{d}})^{1/2}$] regimes. In the kinetic limit (31b), equations (30a) and (30c) yield the "double-streaming" ($\chi_{\text{eff}} = b_0^2 v_e^2 t_{\text{d}}$) and "collisionless" ($\chi_{\text{eff}} = D_{\text{m}} v_{\text{e}}$) regimes, respectively. These four regimes also exist in the quasi-linear limit $R \ll 1$ (KROMMES *et al.*, 1983; ISICHENKO, 1991). Thus, we infer that in the percolation limit $R \gg 1$, in addition to all the regimes pertinent to the quasi-linear limit, there exists a new intermediate regime (IR) given by expression (30b), both in the hydrodynamic and kinetic approximations.

4. DECORRELATION IN A STATIONARY STOCHASTIC MAGNETIC FIELD

In this section we assess the time of test-particle decorrelation t_{d} in a stationary "braided" magnetic field. Among the causes of the decorrelation are either a finite transverse diffusivity $\chi_{\perp}$ (in the collisional case) or a finite gyroradius r_{e} (in the collisionless limit).

While in the quasi-linear limit the decorrelation time has been defined as the time it would take the test particle to leave a magnetic flux tube with initial diameter δ, in the percolation limit the decorrelation occurs when the particle leaves the effective transport region responsible for the anomalous transport. Let us now evaluate the width h of the magnetic cell in various regimes.

In the regime of quick decorrelation, $z_{\text{d}} < \delta/b_0$, where the transport loses its long-correlated features, one concludes that $h \approx \delta$. Otherwise, h is defined by equation (11), where one must substitute for the diameter a either the transverse displacement $\delta(L(z_{\text{d}})/\delta)^{\nu/(\nu+1)}$ in the decorrelation time (in the intermediate regime) or the mixing

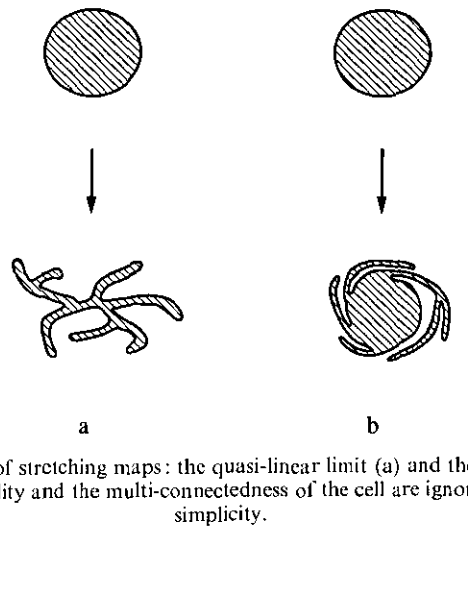

FIG. 2.—Two types of stretching maps: the quasi-linear limit (a) and the percolation limit (b), where the fractality and the multi-connectedness of the cell are ignored for the sake of simplicity.

scale (14) (in the regime of magnetic diffusion). This results in the following width of the magnetic cell, which depends on the decorrelation time t_d:

$$h(t_d) \approx \begin{cases} \delta, & \text{(QD)} \quad (32a) \\ \delta(\delta/b_0 z_d)^{1/(\nu+1)}, & \text{(IR)} \quad (32b) \\ \delta \cdot R^{-1/(\nu+2)}, & \text{(MD)} \quad (32c) \end{cases}$$

where the inequalities are corresponding identical to those of equation (30).

One must now define the quantity t_d as the time it takes the particle to leave a magnetic flux tube whose cross-section $z = 0$ is a magnetic cell whose width given by equation (32). (Imagine Fig. 1 of Part I when in the cross-section $z = 0$ lies not a circle, but the fractal shown in Fig. 1 of this paper.) In a stationary magnetic field and in the collisional case the decorrelation might occur due to the direct cross-field background diffusion with the characteristic time $h^2(t_d)/\chi_\perp$. However, at very small $\chi_\perp$ the electron can decorrelate faster, by first going some distance along the magnetic line, and then diffusing across the smaller width of the magnetic tube $\tilde{h}(z)$ due to the convolution of the magnetic flux tube constructed from the magnetic cell (see Fig. 2). [This effect of stochasticity-driven decorrelation has been pointed out by RECHESTER and ROSENBLUTH (1978) for the quasi-linear limit.] The convolution means the thinning of the tube walls due to the area-preserving stochastic stretching of the field-lines-projected magnetic cell. This effect causes a decrease of the flux-tube thickness $\tilde{h}(z)$, which can be described by the model equation

$$\tilde{h}(z) \approx h \exp(-|z|/l_m), \quad l_m \approx hL_0/\delta. \tag{33}$$

The interconnection of this effect with the stochastic instability, as well as the evaluation of the convolution length l_m for the percolation limit, are discussed in more detail in Appendix B.

Now one can propose an equation for t_d, accounting for both the direct transverse decorrelation and the stochasticity of the magnetic lines:

$$h(t_d) \exp[-z_d(t_d)\delta/(h(t_d)L_0)] = (\chi_\perp t_d)^{1/2}. \tag{34}$$

Together with (31a) this may be readily solved to obtain

$$t_d \approx \begin{cases} \delta^2/\chi_\perp, & \chi_\perp/D_\parallel > 1 \quad \text{(QD)} \quad (35a) \\ (\delta^2/\chi_\perp)(\chi_\perp/D_\parallel)^{1/(\nu+2)}, & 1 > \chi_\perp/D_\parallel > R^{-2} \quad \text{(IR)} \quad (35b) \\ (z_m^2/\chi_\parallel)\ln^2(D_\parallel/\chi_\perp R^2), & R^{-2} > \chi_\perp/D_\parallel. \quad \text{(MD)} \quad (35c) \end{cases}$$

Here for brevity we have introduced the notation $D_\parallel \equiv \chi_\parallel b_0^2$, which means the longitudinal test-particle diffusivity projected on the (x, y) plane.

Let us now turn to the *kinetic limit* $\nu_c t_d < 1$. If, for simplicity, we take the uncertainty in the transverse electron position to be of the order of its gyroradius r_e, as in Part I, then the collisionless decorrelation time t_d should be given by

$$h(t_d) \exp(-z_d(t_d)/l_m) = r_e, \tag{36}$$

together with equations (31b) and (32). At sufficiently small gyroradius, $r_c < h_m \equiv \delta R^{-1/(\nu+2)}$, this yields

$$t_d = (z_m/v_e) \ln (h_m/r_e). \tag{37}$$

The kinetic evaluation (37) smoothly matches the hydrodynamic one in (35c) at $\omega_{B_e}\tau_e = z_m/r_e$.

5. NON-STATIONARY DECORRELATION

Now, let us take $\chi_\perp = 0$, $r_e = 0$ and consider the decorrelation to be the result of only the non-stationarity of the magnetic perturbations $\delta\mathbf{B}(x, y, z, t)$ varying with characteristic frequency ω.

First of all, one can see that the dependence of the perturbations on time simultaneously results in two distinct kinds of decorrelation, which can be referred to as kinematic and dynamic.

On the one hand, percolating magnetic lines evolve in such a way that they reconnect with a preferred shortening of their transverse spread, since smaller contours are more probable. The problem is similar to the one discussed above, concerning the reconnection of contours of $\psi(x, y, z)$ as z is changed. The only difference is that now ψ also depends on time, and the magnetic cells composed of contours of $\psi(x, y, z, t)$ are destroyed (through reconnection) at a fixed z upon the passing of time:

$$t_h = h(t_h)/(\omega\delta). \tag{38}$$

Equation (36) is quite analogous to equation (12) when one makes the substitutions $\delta z \to t_h$, $L_0 \to \omega^{-1}$. The only difference is that now the magnetic cell width h itself depends on the decorrelation time t_h. Thus equation (38) should be solved for the kinematic decorrelation time t_h.

On the other hand, non-stationarity leads to a test particle moving not exactly along the magnetic line, even if the latter does not reconnect. This dynamic decorrelation can be described by equation (3), while accounting for the dependence of $\mathbf{b}$ on time:

$$\mathrm{d}\mathbf{r}_\perp/\mathrm{d}z = \mathbf{b}(\mathbf{r}_\perp, z, t), \tag{39}$$

together with closure condition (31a) or (31b). Let us consider the time-dependence in equation (39) as a small perturbation. Similar to the calculation of Part I, we have:

$$\begin{gathered} \mathrm{d}\mathbf{r}_\perp/\mathrm{d}z = \mathbf{b}(\mathbf{r}_\perp, z, 0) + t(z)\mathbf{b}_1(\mathbf{r}_\perp, z), \\ \mathbf{b}_1(\mathbf{r}_\perp, z) \equiv \partial\mathbf{b}(\mathbf{r}_\perp, z, t)/\partial t|_{t=0} \approx \omega b_0, \\ t(z) = \begin{cases} z^2/\chi_\parallel, & \nu_e t_d > 1 \\ z/v_e, & \nu_e t_d < 1. \end{cases} \end{gathered} \tag{40}$$

The second term on the right-hand side of equation (40) represents a non-correlated (with respect to the first term) slow drift with correlation length $z = \delta/b_0$, which is also the fall-out length of the $\mathbf{b}_1$ correlation function. So the perturbation theory yields the following estimate for the square-average additional displacement $\mathbf{r}_{\perp\omega}$ due to the non-stationarity:

$$\langle \mathbf{r}_{\perp\omega}^2(z) \rangle = \int_0^z \int_0^z \mathrm{d}z'\, \mathrm{d}z''\, t(z')t(z'') \langle \mathbf{b}_1(z')\mathbf{b}_1(z'') \rangle$$
$$\approx \omega^2 b_0^2 \begin{cases} z^2 t^2(z), & z < \delta/b_0 \\ (\delta/b_0) \int_0^z t^2(z')\, \mathrm{d}z', & z > \delta/b_0. \end{cases} \tag{41}$$

The dynamic decorrelation time t_p may now be estimated from an equation similar to (34):

$$h(t_p) \exp\left[-z(t_p)\delta/(h(t_p)L_0)\right] = \langle \mathbf{r}_{\perp\omega}^2(z(t_p)) \rangle^{1/2}. \tag{42}$$

Resolving equations (38) and (42) in each limit [collisional (31a) and collisionless (31b)], we find expressions for t_h and t_p, which are not given here. The true decorrelation time is their minimum: $t_d = \min(t_h, t_p)$.

Comparing the two times t_h and t_p in every interval of parameters, we finally derive the non-stationary decorrelation time. In the hydrodynamic limit $v_c t_d > 1$ the result is

$$t_d \approx \begin{cases} \omega^{-1}, & \text{(QD)} \quad (43a) \\ (\delta^2/D_\parallel)\Omega_{hy}^{-4(\nu+1)/(5\nu+7)}, & \text{(IR)} \quad (43b) \\ (z_m^2/\chi_\parallel) \ln^2 [\Omega_{hy}^{-1} R^{-(5\nu+7)/(2\nu+4)}]. & \text{(MD)} \quad (43c) \end{cases}$$

The corresponding inequalities are:

$$\begin{cases} \Omega_{hy} > 1, & \text{(QD)} \quad (43a) \\ 1 > \Omega_{hy} > R^{-(5\nu+7)/(2\nu+4)}, & \text{(IR)} \quad (43b) \\ R^{-(5\nu+7)/(2\nu+4)} > \Omega_{hy}. & \text{(MD)} \quad (43c) \end{cases}$$

In the kinetic limit $v_c t_d < 1$ we obtain, in a similar way:

$$t_d \approx \begin{cases} \omega^{-1}, & \text{(QD)} \quad (44a) \\ (\delta/v_\parallel)\Omega_{ki}^{-2(\nu+1)/(3\nu+5)}, & \text{(IR)} \quad (44b) \\ (z_m/v_\parallel) \ln [\Omega_{ki}^{-1} R^{-(3\nu+5)/(2\nu+4)}], & \text{(MD)} \quad (44c) \end{cases}$$

$$\begin{cases} \Omega_{ki} > 1, & \text{(QD)} \quad (44a) \\ 1 > \Omega_{ki} > R^{-(3\nu+5)/(2\nu+4)}, & \text{(IR)} \quad (44b) \\ R^{-(3\nu+5)/(2\nu+4)} > \Omega_{ki}. & \text{(MD)} \quad (44c) \end{cases}$$

In equations (43)–(44), the dimensionless frequencies

$$\Omega_{hy} \equiv \omega\delta^2/D_\parallel, \quad \Omega_{ki} \equiv \omega\delta/v_\parallel, \tag{45}$$

have been introduced for the collisional and collisionless cases, respectively. In addition, $v_{\parallel} \equiv v_e b_0$ denotes the projection of the longitudinal electron velocity to the (x, y)-plane.

Note that regardless of the collisionality, in the QD regimes the kinematic decorrelation t_h dominates, while in the other regimes (IR and MD), the dynamic decorrelation time t_p is shorter.

6. EFFECTIVE HEAT CONDUCTIVITY—DISCUSSION OF RESULTS

Formulae (30) and one of the expressions (31a), (35), (43) (in the hydrodynamic limit) or (31b), (37), (44) (in the kinetic limit) solve the problem stated. Among the times of stationary [(35), (37)] and non-stationary [(43), (44)] decorrelation, one should choose the shorter one.

If one knows the main magnetic perturbation parameters b_0, L_0 δ, ω, and the plasma parameters $\chi_{\parallel}$, $\chi_{\perp}$, v_e, v_e, r_e, the effective cross-field heat conductivity can be evaluated with the help of the algorithm shown in Fig. 3.

Let us write down here the expressions for the effective heat conduction in the most obvious limits. Firstly, consider the stationary limit ($\omega = 0$) at $r_e < h_m$. Under such conditions we have

$$\chi_{\text{eff}} \approx \begin{cases} \chi_{\perp}, & \chi_{\perp}/D_{\parallel}, \quad \text{(QD)} \quad (46a) \\ (D_{\parallel}\chi_{\perp})^{1/2}, & 1 > \chi_{\perp}/D_{\parallel} > R^{-2} \quad \text{(IR)} \quad (46b) \\ (D_{\parallel}/R)\ln^{-1}[D_{\parallel}/\chi_{\perp}R^2], & R^{-2} > \chi_{\perp}/D_{\parallel} > (r_e/b_0 z_m)^2 \quad \text{(MD)} \quad (46c) \\ D_m v_e, & (r_e/b_0 z_m)^2 > \chi_{\perp}/D_{\parallel}. \quad \text{(MD)} \quad (46d) \end{cases}$$

The first three regimes (46a–c) are hydrodynamic while the last one (46d) is kinetic.

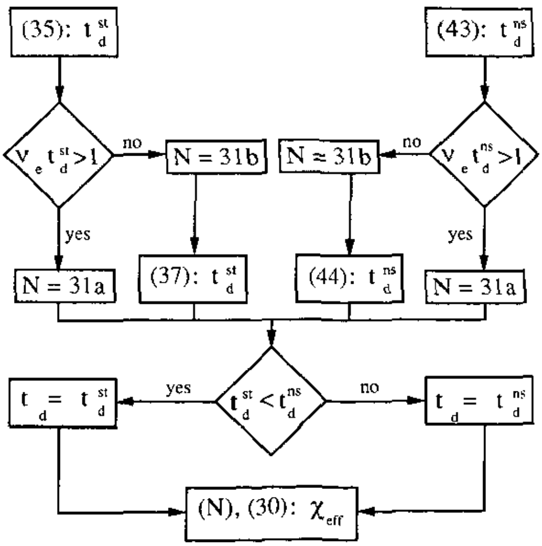

FIG. 3.—Flow chart of effective heat conductance evaluation. The formula numbers to be used are given in parentheses.

In the case of strictly two-dimensional magnetic perturbations ($R = \infty$), the result is given by expression (46b). The same estimate has been obtained by KADOMTSEV and POGUTSE (1979) in two different ways—one of them by using the DYKHNE (1971) method incorrectly. The essence of this misunderstanding and the application of the Dykhne technique are discussed in Appendix C.

It is interesting that expressions (46b) and (46c), related to the percolation limit, do not include the percolation exponent ν, and hence could be obtained using a simplified independent approach, as demonstrated KADOMTSEV and POGUTSE (1979). [The only difference between (46c) and their result lies in the logarithmic denominator.] However, the mixing length a_m does include ν (see Appendix C).

Similar to the quasi-linear limit, non-trivial quick decorrelation regimes, i.e. QD regimes with $\chi_{\rm eff} \gg \chi_\perp$, such as "fluid" ($\chi_{\rm eff} = D_\parallel$) and "double-streaming" regimes, become possible only in a non-stationary stochastic magnetic field. Taking $\chi_\perp = 0$, $r_e = 0$, and accounting only for the finite frequency ω of the perturbations, one obtains from equations (30), (43) and (44):

$$\chi_{\rm eff} \approx \begin{cases} D_\parallel, & \text{(QD)} \quad (47a) \\ D_\parallel \Omega_{hy}^{2(\nu+2)/(5\nu+7)}, & \text{(IR)} \quad (47b) \\ (D_\parallel/R)\ \ln^{-1}\left[\Omega_{hy}^{-1} R^{-(5\nu+7)/(2\nu+4)}\right], & \text{(MD)} \quad (47c) \end{cases}$$

for the collisional limit $\nu_e t_d > 1$; and

$$\chi_{\rm eff} \approx \begin{cases} v_\parallel^2/\omega, & \text{(QD)} \quad (48a) \\ \delta v_\parallel \Omega_{ki}^{2/(3\nu+5)}, & \text{(IR)} \quad (48b) \\ D_m v_e, & \text{(MD)} \quad (48c) \end{cases}$$

in the collisionless case $\nu_e t_d < 1$, where the inequalities for (47) and (48) are the same as those in expressions (43) and (44), respectively.

One can follow the transition between different regimes when the characteristic frequency ω varies. Figure 4 demonstrates this transition for the case $\delta/b_0 < \lambda_e < z_m$, where $\lambda_e = v_e/\nu_e$ is the mean-free-path of electrons.

In conclusion, we restate the key points of the analysis:

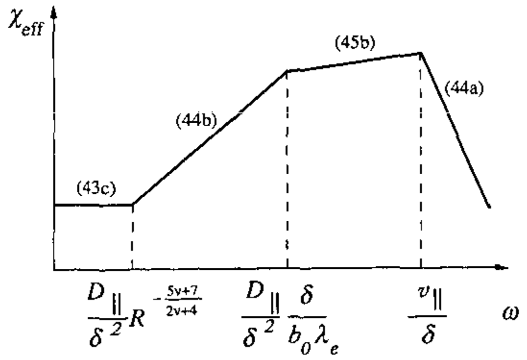

FIG. 4.—The dependence of $\chi_{\rm eff}$ on ω at $R \gg 1$, $\delta/b_0 \ll \lambda_e \ll z_m$.

(i) The theory of anomalous transport in a "braided" magnetic field in the strong turbulence limit $R \gg 1$, which is opposite to the quasi-linear limit, must use the percolation-theory methods.
(ii) In the percolation limit $R \gg 1$, excluding quick decorrelation regimes, the main transport is concentrated on fractals, consisting of a small fraction of percolating magnetic lines.
(iii) In addition to all the quasi-linear regimes or their direct analogs, in the percolation limit there arise a number of new intermediate regimes of anomalous electron heat conductivity.

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APPENDIX A: EFFECT OF COMPRESSIBILITY OF TRANSVERSE MAGNETIC PERTURBATION

The compressibility of **b** takes place only in the case of a non-zero longitudinal component of the magnetic perturbation: $b_z(x, y, z) \equiv \delta B_z/B_0 \neq 0$. Then the transverse magnetic component can be written in the form

$$\mathbf{b} = \nabla_\perp \psi(x, y, z) \times \hat{z} + \nabla_\perp \varphi(x, y, z). \tag{A1}$$

Taking the divergence of equation (A1), we obtain the relation for φ:

$$\nabla_\perp^2 \varphi = -\partial b_z/\partial z. \tag{A2}$$

The field line motion due to the magnetic perturbation (A1) consists of two parts: the incompressible motion $\nabla_\perp \psi \times \hat{z}$ (approximately along the isolines of ψ), plus a small drift on account of the potential correction. The drift part of the displacement can be calculated as

$$\mathbf{r}_{\perp\varphi}(z) = \int_0^z \nabla_\perp \varphi(\mathbf{r}_\perp(z'), z')\,\mathrm{d}z', \tag{A3}$$

where $\mathbf{r}(z')$ is the solution of the non-perturbed equation (3) (i.e. at $\varphi = 0$). Since the quantity φ changes its sign randomly along the isolines of ψ, the drift (A3) can be described in terms of diffusion: $r_{\perp\varphi}(z) \approx (D_\varphi z)^{1/2}$, where the diffusion coefficient is the product of characteristic speed and the correlation length: $D_\varphi \approx (\varphi/\delta)(\delta/b_0)$. Using the estimate $\varphi \approx b_z\delta^2/L_0$, which follows from (A2), we obtain the additional displacement caused by the compressibility correction:

$$r_{\perp\varphi} \approx (zb_z\delta/R)^{1/2}. \tag{A4}$$

Comparing this quantity with the minimum width of the magnetic cell h_m (32c), at z, corresponding to the maximum longitudinal mixing length z_m (15), we arrive at the following condition, under which one may neglect compressibility effects:

$$b_z/b_0 < R^{-1/(\nu+2)}. \tag{A5}$$

Inequality (A5) is the criterion for the neglect of the compressibility of **b** in MD regimes of anomalous transport. In IR regimes condition (A5) is sufficient, however not necessary, since in that case the effective transport width h given by equation (32b) is greater than h_m, while the longitudinal mixing length z_a is less than z_m. In QD regimes the compressibility of **b** is irrelevant and does not affect the transport.

Note that in a strong magnetic field the longitudinal perturbation of **B** is energetically much more expensive than the transverse one and hence must be much smaller. This makes condition (A5) not too restrictive.

APPENDIX B: CONVOLUTION OF MAGNETIC FLUX TUBE

The map

$$\mathbf{r}_\perp(0) \to \mathbf{r}_\perp(z), \tag{B1}$$

given by the initial-value problem solution of equation (3), may be thought of as an incompressible, and consequently, a Hamiltonian one. The corresponding Hamiltonian $\psi(x, y, z)$, depending on "time" z, admits stochastic behavior; moreover, such a behavior is typical for a generic Hamiltonian (c.f. ARNOLD, 1978). It means that every curve in the phase space (x, y), consisting of points evolving according to the equation of motion (3), elongates in time* exponentially, as every two close points exponentiate from each other. The mean growth rate of this stochastic instability (the Kolmogorov entropy) has been calculated in the quasi-linear limit $R = v/(\delta\omega) \ll 1$ by KROMMES (1978), RECHESTER *et al.* (1979) and KROMMES *et al.* (1983) for the case of a strong shear. For the opposite limit without shear the Kolmogorov entropy at $R \ll 1$ has been estimated by KADOMTSEV and POGUTSE (1979):

$$\gamma_s \approx \omega R^2, \quad R \ll 1. \tag{B2}$$

In the percolation limit $R \gg 1$ γ_s has been calculated by GRUZINOV *et al.* (1990):

$$\gamma_s \approx \omega R^{1/2} \ln R, \quad R \gg 1. \tag{B3}$$

Rigorously speaking, result (B3) is not the Kolmogorov entropy, being the mean growth rate of the exponentiation, but rather the topological entropy, or maximum growth rate, defining the elongation of a liquid curve.

The decorrelation of test particles in the magnetic field is not directly related to the rate of stretching of a curve, but rather to the convolution of a flux tube constructed from a magnetic cell. This effect is connected with the evolution of the characteristic width $\tilde{h}(t)$ of the Lagrangian convection cell, which in hydrodynamic terms corresponds to the magnetic cell. This width can be defined as the shortest distance between a point, situated at $t = 0$ somewhere in the middle of the cell [i.e. $\tilde{h}(0) \approx h$], and the cell's boundary. The "Lagrangian convection cell" represents the flow-driven image of the convection cell (effective transport region).

There is a definite connection between stochastic instability of orbits and the Lagrangian stretching of the convection cell. However, this connection is quite different in the quasi-linear and percolation limits. The difference begins with the appearance of the cells: while in the quasi-linear limit a circle with diameter δ can be considered as a magnetic cell (if there is any sense in this notion at all), but in the percolation limit this is a fractal a_m-type cell of the contours of vector potential ψ (see Fig. 1). Furthermore, even for similar 2-D domains one can imagine two kinds of area-preserving maps with exponentially-elongated

* In this Appendix, for the sake of clarity, we take $z \to t$, $\mathbf{b} \to \mathbf{v}$, $L_0 \to \omega^{-1}$, thus transfering the magnetic-line problem to the passive scalar problem in the random 2-D incompressible, weakly non-stationary flow $\mathbf{v}(x, y, t) = \nabla\psi(x, y, t) \times \hat{z}$, varying in time with a small characteristic frequency $\omega \ll v/\delta$. In this representation the magnetic line diffusion corresponds to the "turbulent diffusion" in the flow.

curves. The type-I map stretches all the region at once (see Fig. 2a). The map of the second type affects for some time only those parts of the domain which are very close to its initial boundary (see Fig. 2b). One can easily understand that neither of the two contradicts a global stochastic instability. At the same time in the first case the characteristic width $\tilde{h}(t)$ decreases at a rate inversely proportional to the perimeter, and in the second case it decreases much slower.

Type-I stretching takes place in the quasi-linear limit due to the fast variation of the velocity field, which gives rise to the destruction of the flow "memory". Type-II stretching is characteristic of the percolation limit. In the low-frequency limit $R \gg 1$, the elongation of a Lagrangian curve results from its hooking the saddle points of the flow and its dragging in the channels between the separatrices. (A separatrix means an "eight-like" stream line coming through a saddle.) The exponentiation with rate (B3) occurs as a result of the reconnection of adjacent separatrices, moving with a velocity of the order of $\omega\delta$, due to the flow non-stationarity. The inverse growth rate (B3) corresponds to the time it takes a separatrix of the length L to pass the distance δ^2/L to the nearest separatrix, under an optimum choice of L (Gruzinov *et al.*, 1990). Yet, during this small time $t_s = \delta/(\omega L)$ the velocity field remains almost unchanged, as the convection cell contains $h/(\delta^2/L) \approx a/\delta \gg 1$ separatrices. The life-time of the convection cell, corresponding to the intersection of the most remote separatrices, is much longer:

$$t_h \approx h/(\omega\delta) \gg t_s. \tag{B4}$$

(Compare with equation (12).) This means that during time t_s the Lagrangian convection cell is nearly unchanged except for narrow channels of width δ^2/L in the vicinity of its boundaries. As the separatrices of the convection cell keep on reconnecting, the Lagrangian convection cell grows new exponentiating "whiskers" (see Fig. 2b). Finally, near the end of the life-time (B4) all the domain is subject to the intensive stretching with the rate given by equation (B3).

Hence, the life-time (B4) of the convection cell is also the characteristic time of the Lagrangian convection cell stretching. In terms of three-dimensional stationary magnetic fields the time (B4) corresponds to the following length l_m of the convolution of a flux tube constructed from a magnetic cell:

$$l_m = L_0 h/\delta, \tag{B5}$$

where h means the width of the magnetic cell. However, as it is seen from above, this process is rather complicated and has its own stages. Consequently, formula (33) is a model one, and the results (35c), (43c), (44c), (46c) following from it are valid to a logarithmic accuracy only.

APPENDIX C: EFFECTIVE TRANSPORT IN TWO-DIMENSIONAL ANISOTROPIC RANDOM MEDIA

Let us consider a two-dimensional anisotropic medium in which the direction of the anisotropy $\mathbf{n}(x, y)$ is a function of the coordinates, $|\mathbf{n}| = 1$. Let us suggest that along this direction the electric conductivity (or heat conductivity, diffusivity, etc.) equals σ_1 while in the perpendicular direction it is equal to σ_2. So, the local Ohm's law takes the form

$$\mathbf{j} = \sigma_1 \mathbf{E}_{\parallel} + \sigma_2 \mathbf{E}_{\perp}, \tag{C1}$$

$$\mathbf{E}_{\parallel} = \mathbf{n}(\mathbf{E}\mathbf{n}), \quad \mathbf{E}_{\perp} = \mathbf{E} - \mathbf{E}_{\parallel}, \tag{C2}$$

$$(\nabla \mathbf{j}) = 0, \quad \nabla \times \mathbf{E} = 0. \tag{C3}$$

Suppose further that the medium is a self-averaged one, and the mean conductivity is isotropic; i.e. from (C1)–(C3) it follows that

$$\langle \mathbf{j} \rangle = \sigma_{\text{eff}}(\sigma_1, \sigma_2; \mathbf{n}(x, y)) \langle \mathbf{E} \rangle. \tag{C4}$$

Here the angular brackets mean space-averaging over a domain large compared to some mixing length a_m.

Using the ansatz $\mathbf{j}' = C_1 \hat{z} \times \mathbf{E}$, $\mathbf{E}' = C_2 \mathbf{j} \times \hat{z}$ and comparing (C4) with the resulting "Ohm's law" for $\mathbf{j}'$, $\mathbf{E}'$, Dykhne (1971) has shown that the effective conductivity satisfies the relation

$$\sigma_{\text{eff}}(\sigma_1, \sigma_2; \mathbf{n}(x, y)) \sigma_{\text{eff}}(\sigma_2, \sigma_1; \mathbf{n}'(x, y)) = \sigma_1 \sigma_2, \tag{C5}$$

where $\mathbf{n}'(x, y) = \hat{z} \times \mathbf{n}(x, y)$ is the perpendicular direction field. From equation (C5) it follows that the effective conductivity of a two-dimensional polycrystal with random directions of the main axes of crystallites [$\mathbf{n}(x, y)$ is uniform inside every crystallite and discontinuous on boundaries between them] is equal to $(\sigma_1 \sigma_2)^{1/2}$. In that case the two fields $\mathbf{n}$ and $\mathbf{n}'$ are statistically equivalent, and σ_{eff} is an even function of σ_1 and σ_2, which proves the Dykhne (1971) result.

In a 2-D magnetic field (2) (with $\partial/\partial z = 0$) the problem of heat conductivity is equivalent to (C1)–(C3), the field of anisotropy direction being given by a smooth function

$$\mathbf{n} \propto \mathbf{b}(x, y) = \nabla\psi \times \hat{z}. \quad \text{(C6)}$$

Further, one should put

$$\sigma_1 = (\chi_\parallel \mathbf{b}^2 + \chi_\perp)/(1+\mathbf{b}^2), \quad \sigma_2 = \chi_\perp. \quad \text{(C7)}$$

In such a problem the fields **n** and $\mathbf{n}' \propto \nabla\psi$ are not statistically equivalent, as one of them is proportional to a solenoidal field and the other to a potential one. At first glance, this makes it impossible to apply the Dykhne method to this problem in the way that has been done by Kadomtsev and Pogutse (1979). Nevertheless, their result

$$\chi_{\text{eff}} = (\sigma_1\sigma_2)^{1/2} \approx b_0(\chi_\parallel\chi_\perp)^{1/2} = (D_\parallel\chi_\perp)^{1/2} \quad \text{(C8)}$$

turned out to be correct, which is due to the following simple observation: *the two media* $(\sigma_1, \sigma_2; \mathbf{n})$ *and* $(\sigma_2, \sigma_1; \mathbf{n}')$ *are in fact identical.* Consequently, regardless of the statistical equivalence of the two fields **n** and **n**′ the two factors in the left-hand side of equation (C5) are equal.

Thus, when the effective conductivity of a two-dimensional locally-anisotropic self-averaging medium is isotropic, then it equals exactly

$$\sigma_{\text{eff}} = (\sigma_1\sigma_2)^{1/2}. \quad \text{(C9)}$$

It is rather instructive to obtain an assessment of the exact result (C9) in another way together with the evaluation of the mixing length a_m. Here it is more convenient to argue in terms of diffusivity. Let σ_1 be much greater than σ_2. The characteristic **b** lines with length L responsible for the effective transport are defined by that in the mixing time $\tau_m = L^2/\sigma_1$ needed for the longitudinal particle diffusion, the particle leaves the percolation cell width h on account of the transverse diffusivity σ_2:

$$\tau_m \approx L^2/\sigma_1 \approx h^2/\sigma_2. \quad \text{(C10)}$$

Taking (10) and (11) into account, this yields the mixing length

$$a_m = \delta(\sigma_1/\sigma_2)^{\nu/(2\nu+4)}. \quad \text{(C11)}$$

The effective diffusion is defined according to $\sigma_{\text{eff}} \approx F(a_m)a_m^2/\tau_m$, where $F(a_m) = Lh/a_m^2$ is the share of the percolating **b** lines. When taking (C10) into consideration, this results in formula (C9).

Note that the feature of the large mixing lengths ($a_m \gg \delta$) is typical for percolation-like transport problems and at shorter scales the transport processes are non-diffusive.